Boundary Value Problems.

How to use published results ~~Res~~ -

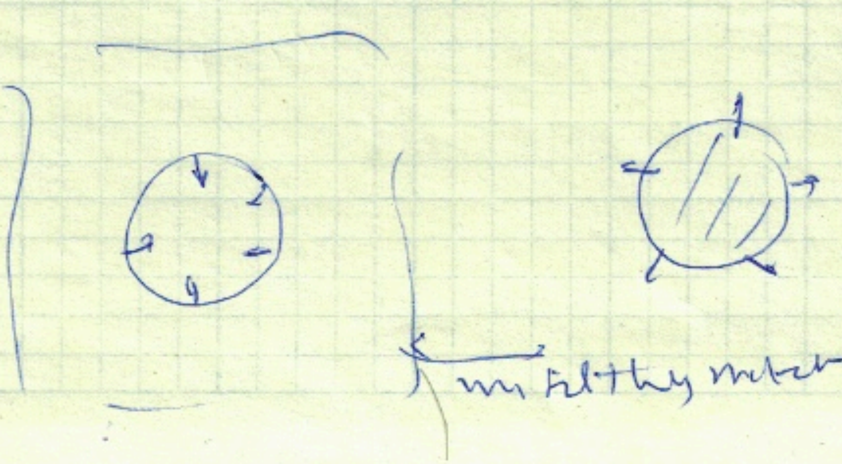
~~Exp~~
Example : inclusion within a matrix.

Tension at the interface

Ceramic within metal matrix

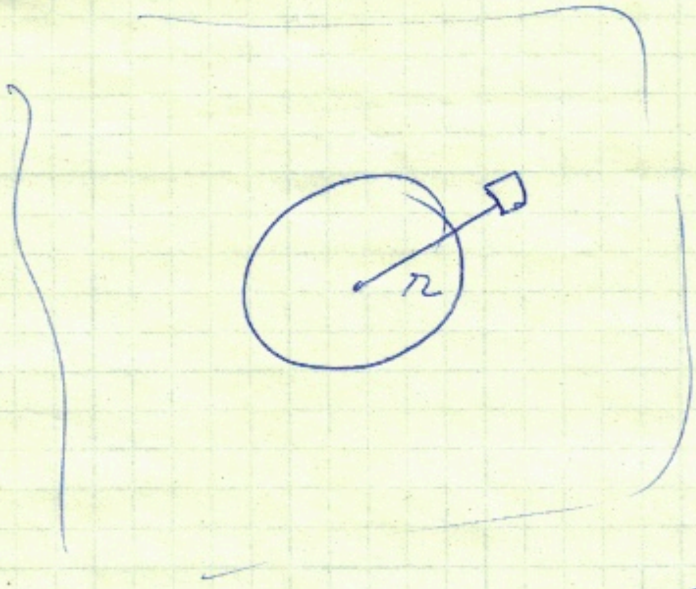
ΔT

THE METHOD



misfitting metal

 MDMS



A
B are consten

$$U_r = \cancel{Ar} + \frac{B}{r^2} \quad \text{— (1)}$$

$$\sigma_{rr} = -\frac{2E}{(1+\nu)r^3}B + \cancel{\frac{E}{(1-2\nu)}A} \quad \text{— (2)}$$

$$\sigma_{\theta\theta} = \frac{E}{(1+\nu)r^3}B + \cancel{\frac{E}{(1-2\nu)}A} \quad \text{— (3)}$$

$r \to \infty \quad U_r = 0 \quad \sigma_{rr} + \sigma_{\theta\theta} = 0$

$\therefore \; A = 0$

$\sigma_{rr} = -\sigma_{\theta\theta}$ pure shear.

Assume inclusion is rigid

R = inclu rad. $\quad U_r(r=R) = R\,\Delta T\alpha$.

$$R\,.\,\Delta T\alpha = B$$

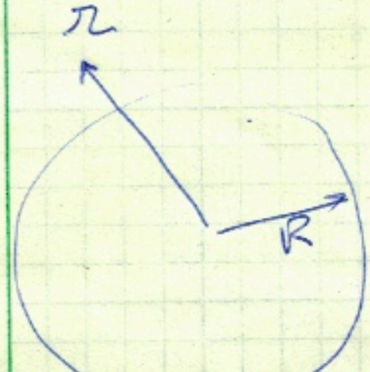
$$\sigma_{rr} \overset{2\times}{=} \sigma_{\theta\theta} = \frac{2E \times R\Delta T\alpha}{(1+\nu)R^3}$$

$$\sigma_s \propto \frac{1}{R^2}\cdot\frac{E}{(1+\nu)}\cdot\Delta T\cdot\alpha. \quad \frac{\sigma_s\,max}{r=R}$$

The stress tensor
$$\begin{bmatrix} \sigma_{rr} & & \\ & \sigma_{\theta\theta} & \\ & & \sigma_{\theta\theta} \end{bmatrix}$$

where (assuming A = 0 in Eqns 2 & 3 on the previous page)

A = 0 because at $r \to \infty$ (we assume a very large matrix) $\sigma \to 0$



$$\sigma_{rr} = -\frac{2E}{(1+\nu)r^3} \cdot B \quad \text{— (4)}$$

$$\sigma_{\theta\theta} = \frac{E}{(1+\nu)r^3} \cdot B \quad \text{— (5)}$$

$$u_r = \frac{B}{r^2} \quad \text{— (6)}$$

R = radius of the hole

Note that if u_r is +ve then σ_{rr} is compressive and $\sigma_{\theta\theta}$ is tensile

Sum of the diagonal terms is the hydrostatic tension, σ_H:

$$\sigma_H = \sigma_{rr} + 2\sigma_{\theta\theta}$$

From (4) & (5) we note that

$$\sigma_H = 0$$

that is, the <u>state of stress</u>, in the matrix is pure shear even though the hole is uniformly expanded or compressed, that is, the matter placed within the hole is in hydrostatic tension (u_2 +ve) or hydrostatic compression (u_2 is -ve).

In the following derivations we consider cases where the inclusion & the matrix have a different coefficient of thermal expansion ($\Delta\alpha$) and stresses are generated by changing the temperature (ΔT). We shall consider the case where (i) ΔT is +ve and where

$\Delta\alpha = (\alpha_{matrix} - \alpha_{inclusion})$ is positive.

Therefore we can note that when the temperature is increased the matrix hole will want to expand more than it can and the inclusion will be placed under hydrostatic tension.

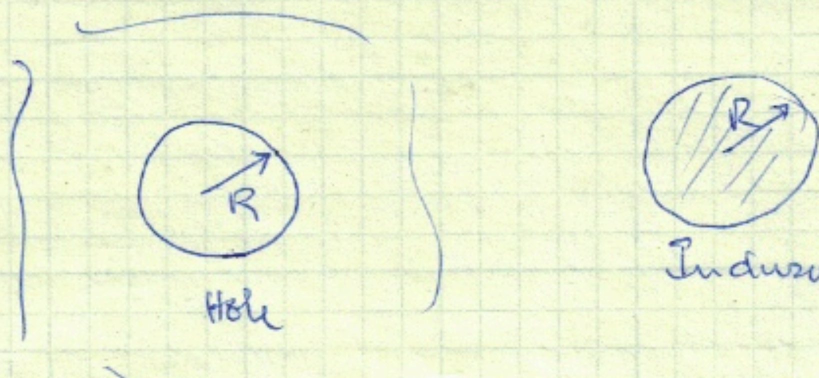
Limiting Case (Rigid inclusion)

E_M is finite

E_I is ∞ that is the inclusion is rigid.

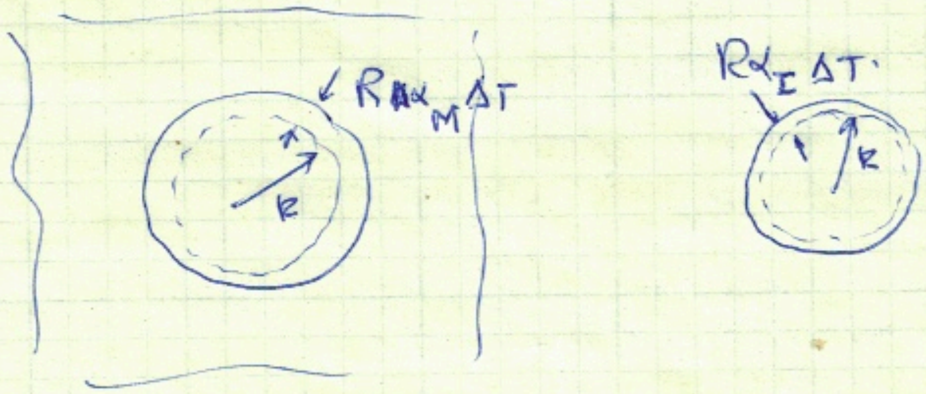
Method

Step I :



Now increase the temperature by ΔT

$(\alpha_{matrix} > \alpha_{inclu}):$



Now the misfit displacement

$$\Delta u = -(u_{hole} - u_{inch})$$

$$= -(R\alpha_M\Delta T - R\alpha_I\Delta T)$$

$$\Delta u = -R\,\Delta\alpha\,\Delta T \quad \text{—— (7)}$$

Step II

Now we apply a tensile stress $(\sigma_{rr}(r=R))$ to the inside surface of the hole, and a tensile stress $(\sigma_{rr}(r=R))$ equal to the stress on the matrix hole, to the surface of the inclusion, and calculate the displacements.

Since the inclusion is assumed to be rigid

$$u_I(R) = 0$$

$$u_M(R) = -R\,\Delta\alpha\,\Delta T \quad (= \Delta u) \quad \text{—— (8)}$$

Step III apply Eqns ④, ⑤ & ⑥

From Eq. (6)

$$U_M(R) = -R\,\Delta\alpha\,\Delta T = -\frac{B}{R^2} \quad \text{— ⑨}$$

which yields the expression for the unknown B

$$\therefore\ B = -R^3\,\Delta\alpha\,\Delta T \quad \text{—— ⑩}$$

Step IV

- Calculate the hydrostatic stress in the inclusion
- Calculate the shear stress in the matrix
- Calculate the stress (traction) across the interface.

$$\sigma_I^* = \sigma_{rr}(r=R).$$

- Hydrostatic stress in the inclusion is equal to zero since the inclusion is rigid.

- The stress Tensor in the matrix

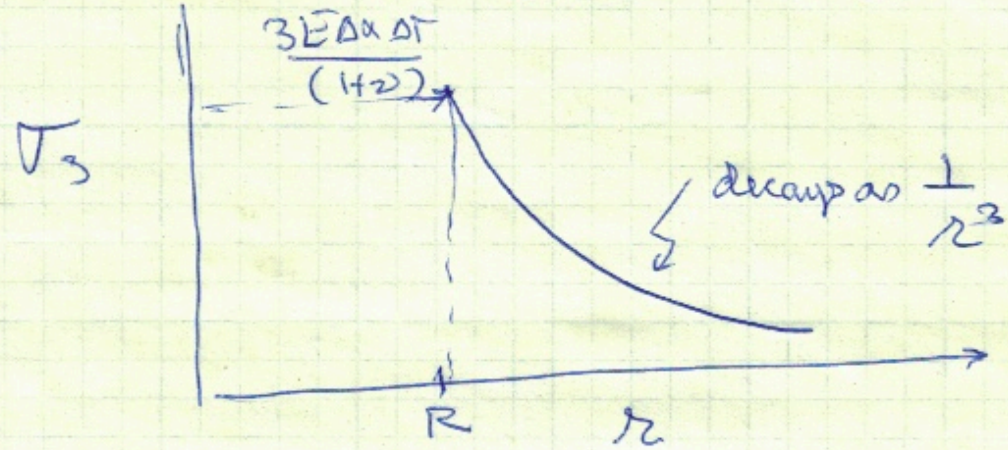
Eq ④ + ⑩

Eq. ⑤ + ⑩

$$\begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{bmatrix} = \begin{bmatrix} +\frac{2E}{(1+\nu)r^3} R^3\Delta\alpha\Delta T & & \\ & -\frac{E}{(1+\nu)r^3} R^3\Delta\alpha\Delta T & \\ & & -\frac{E}{(1+\nu)r^3} R^3\Delta\alpha\Delta T \end{bmatrix}$$

Shear Stress in the matrix $\sigma_s = \left| \frac{(\sigma_1 - \sigma_2)}{2} + \frac{(\sigma_1 - \sigma_3)}{2} \right|$

$$\sigma_s = \frac{3E\,\Delta\alpha\,\Delta T}{(1+\nu)} \cdot \frac{R^3}{r^3} \quad \text{— (11)}$$



Note that the solution should depend only on the shear modulus of the matrix. This is evident for

$$\frac{E}{2(1+\nu)} = G$$

$$\therefore \quad \sigma_s = 6G\,\Delta\alpha\,\Delta T\,R^3 \cdot \frac{1}{r^3} \quad \text{— (11a)}$$

Note: the units are okay.

Tensile stress at the interface

$$\sigma^*_{rr}(r=R) = \left[\frac{6G}{r^3} \cdot R^3\,\Delta\alpha\,\Delta T\right]_{r=R}$$

$$= 6G\,\Delta\alpha\,\Delta T \quad \text{(tensile stress)}$$

Case II Inclusion is compressible

The important elastic constants since the inclusion deforms hydrostatically while the matrix deforms in shear.

B_I = Bulk modulus of the inclusion

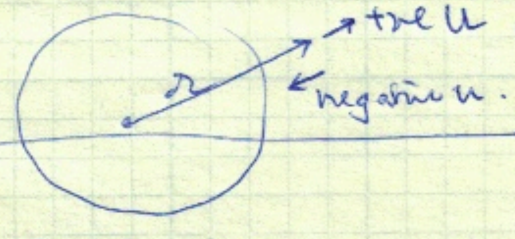
G_M = shear modulus of the matrix

We build upon the Equations from the rigid inclusion analysis.

Eq. (7) remains unchanged.

$$\Delta u = -R\Delta\alpha\Delta T \quad — \quad (12)$$

Note how the sign for u is defined



However Eq (8) changes.

$$u_I(R) = R\alpha_I\Delta T$$

$$u_M(R) = R\alpha_M\Delta T$$

(assume that $\alpha_M > \alpha_I$. Interface in Tension)

$$\therefore \Delta u = u_I(R) - u_M(R)$$

$$= -R\Delta\alpha\Delta T \quad \text{where} \quad — \quad (13)$$

$$\Delta\alpha = \alpha_M - \alpha_I$$

(Same as for the rigid inclusion)

The next step is to apply an interface stress (tensile) defined as

$$\sigma_R^*$$

and calculate the elastic displacements on the ~~I~~ inclusion and the hole such that the sum of these displacements becomes equal to Eq. (13), that is, the elastic forces produce displacements that close the gap produced by a difference in the thermal expansion of the inclusion and the matrix.

u_I^* is the displacement on the inclusion (it is a positive quantity)

u_M^* is the displacement on the surface of the hole – it is a negative quantity.

The total displacement $= u_I^* - u_M^* = -R\,\Delta\alpha\,\Delta T$ (as given by Eq. (13))

$\underline{u_I^*}$ $\qquad \sigma_R^* =$ ~~p~~ hydrostatic tension exerted on the inclusion.

$$\therefore \quad \sigma_R^* = \frac{\Delta V}{V} \cdot B_I \qquad \text{—— (14)}$$

B_I ← Bulk modulus of the inclusion.

$$\frac{\Delta V}{V} \Leftarrow \frac{\frac{4}{3}\pi\left(R^{*3} - R^3\right)}{\frac{4}{3}\pi}$$

$$\frac{d}{dR}$$

$$\frac{\Delta V}{V} = \boxed{\frac{\sigma_R^*}{B_I}} = \frac{\frac{4}{3}\pi(R^{*3} - R^3)}{\frac{4}{3}\pi R^3}.$$

R - initial radius

R^* - radius after hydrostatic tension.

$$= \left(\frac{R^*}{R}\right)^3 - 1.$$

$$R^* = u_I^* + R.$$

$$\left(\frac{R^*}{R}\right)^3 = \left(\frac{u_I^* + R}{R}\right)^3$$

$$= \left(1 + \frac{u_I^*}{R}\right)^3$$

$$\simeq \left(1 + \frac{3u_I^*}{R}\right)$$

$$\frac{\Delta V}{V} = \left(\frac{R^*}{R}\right)^3 - 1 = 1 + \frac{3u_I^*}{R} - 1 = \frac{\sigma_R^*}{B_I} \text{ (from Eq. 14)}$$

$$u_I^* = \frac{\sigma_R^*}{B_I} \times \frac{R}{3} \qquad (15)$$

Now U_M^*

Need a relationship between σ_R^* and U_M^* which we get by combining Eqn (4) & Eq (6).

$$\underbrace{U_M^* = \frac{B}{R^2} = -\frac{\sigma_R^*(1+\nu)R^3}{2E_M} \times \frac{1}{R^2}}_{\text{Eliminate } B \text{ from Eq } 4 \ \& \ 6}$$

write $G_M = \dfrac{E_M}{2(1+\nu)}$

$$U_M^* = -\frac{\sigma_R^*}{4G_M} \cdot R \qquad \text{— (16)}$$

Now $\Delta u = -R\Delta\alpha\Delta T = U_I^* - U_M^*$.

$$R\Delta\alpha\Delta T = \sigma_R^* \times R\left[\frac{1}{3B_I} + \frac{1}{4G_M}\right]$$

$$\sigma_R^* = \Delta\alpha\Delta T \cdot \frac{12B_IG_M}{4G_M + 3B_I} \qquad \text{— (17)}$$

(Note σ is independent of the size of the inclusion

Full solutions are obtained by combining Eq (17) & Eq (4) at $r \to R$ to obtain Ⓑ

& as applied to the matrix

$$G_M = \frac{E_M}{2(1+\nu)}$$

$$\underbrace{-\frac{4 G_M Ⓑ}{R^3}}_{Eq(4)} = \frac{\Delta\alpha \Delta T \; 12 B_I G_M}{4 G_M + 3 B_I}.$$

$$Ⓑ = -\left(\frac{3 B_I}{4 G_M + 3 B_I}\right) \times \Delta\alpha \Delta T R^3.$$

$$\sigma_{rr}^{Matrix} = \Delta\alpha \Delta T \frac{R^3}{r^3} \times \frac{3 B_I \times 4 G_M}{4 G_M + 3 B_I}.$$

$$\sigma_{\theta\theta}^{Matrix} = -\frac{\sigma_{rr}^{Matrix}}{2}$$

$$\sigma_s^{Matrix} = \sigma_{rr}^{Matrix} - \sigma_{\theta\theta}^{Matrix}$$

$$= \frac{3}{2} \Delta\alpha \Delta T \frac{R^3}{r^3} \times \frac{3 B_I \times 4 G_M}{4 G_M + 3 B_I}.$$

Question: $\sigma_s^{Matrix}(r=R) >$ yield stress of the matrix assumed to be metal

$$\left(\sigma_s^{m}\right)_{max} \quad r \to R$$

$$\hookrightarrow = \underbrace{\frac{3}{2} \cdot \Delta\alpha\Delta T}_{\text{strain}} \cdot \frac{3B_I \cdot 4G_M}{4G_M + 3B_I}$$

units of Elastic modul.

$$\alpha_{Cu} = 17 \times 10^{-6} \text{ strain}/°C$$

$$\alpha_{Al_2O_3} = 4.5 \times 10^{-6} \text{ str}/°C.$$

yields when

$$\frac{3}{2}\Delta\alpha\,\Delta T \approx 10^{-3} \quad (0.1\%).$$

$$\Delta T = \frac{2}{3} \frac{\times 10^{-3}}{12.5 \times 10^{-6}} \,°C$$

$$= \frac{2}{37.5} \times 10^{+3} \,°C.$$

$$= \frac{1000}{20} = \underline{\underline{50°C}}$$

$$\alpha_{ZrO_2} \approx 10 \times 10^{-6} \text{ strain}/°C.$$

Interface tensile stress

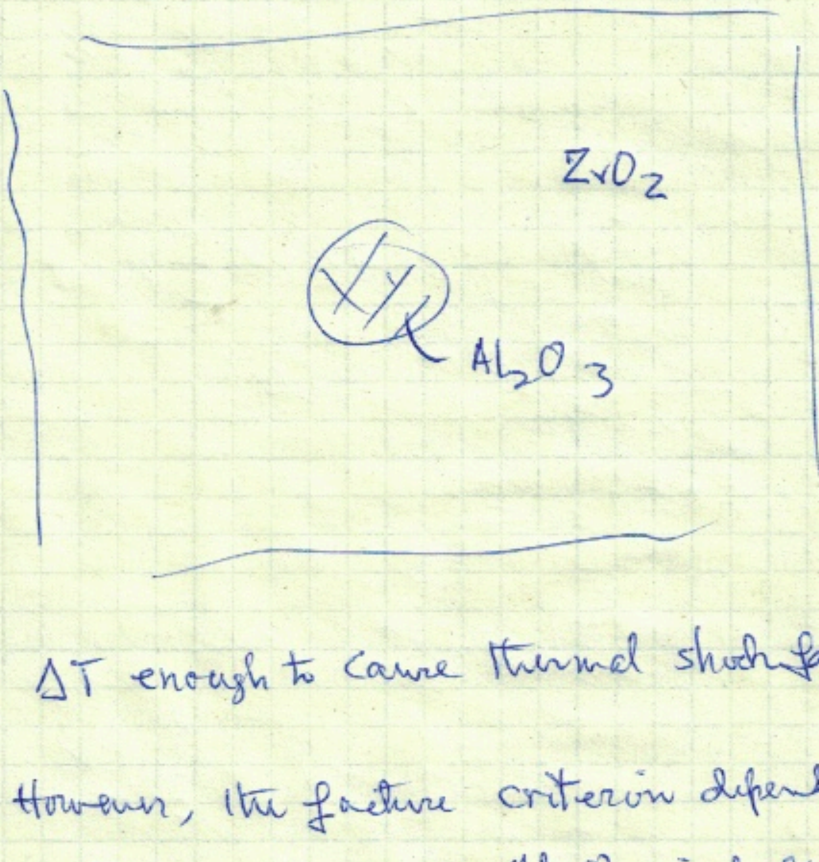
$$\sigma''_{Interf} = \Delta\alpha\Delta T \cdot \frac{3B_I 4G_M}{4G_M + 3B_I}$$

Assume fracture at 1% elastic strain

$$\sigma^*_{int} = 0.01 \times (\text{Modulus})$$

$$\Delta\alpha \Delta T = 0.01.$$

$$\Delta T_{Frac} = 500^\circ C$$



ΔT enough to cause thermal shock failure.

However, the fracture criterion depends to the particle size of Al_2O_3 inclusions